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Force equation of the large-scale structure of the Universe

Martín López-Corredoira

Instituto Astrofísico de Canarias, E-38200 La Laguna, Tenerife, Spain

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Abstract. This article presents a statistical-mechanical treatment of a relationship (the force equation) between the gravitational potential for two particles and the correlation functions in a relaxed distribution of particles with different masses. This relationship is used in the case of galaxies interacting through a Newtonian potential in an Universe in expansion, i.e. the large-scale distribution of galaxies.

By applying this equation and from the observed two-point correlation function for galaxies as a -1.8 exponent power law, I derive the approximate dependence of a mass-mass correlation function as a -2.8 exponent power law, i.e. I infer that mass is more correlated than galaxies at short distances, when the distribution is considered as relaxed.

1. Introduction

In a physical system consisting of a collection of particles, information can be obtained from the way in which the particles are distributed in space. For a Poissonian distribution in equilibrium there is no correlation among the the particles, the position of each one being independent on the position of the others. This distribution sets the particles randomly in space with equal probability for all the positions. In contrast, when the distribution deviates from Poissonian, the correlations among particles are not null; instead, the particles will be mutually interacting.

The relationship between the correlations of the distribution and the interaction force is called the *force equation* and can be obtained under a certain hypothesis (also called the *cosmic virial theorem* in the literature relating to large-scale structure; Peebles 1976). The development of statistical-mechanical theory of liquids has allowed this relationship to be achieved in the specific case where the particles are all equal, point-like and in Boltzmann equilibrium, which is quite common for classical particles in thermodynamic equilibrium (see for example Goodstein 1975, March & Tosi 1976). In general, statistical-mechanical tools are applied to obtain the distribution of particles (molecules on the atomic scale) from an assumed form of the interaction (the Lennard-Jones interaction among molecules).

Since these statistical applications can be developed in environments other than on the atomic scale, my purpose is to derive a few relationships that will be useful in cosmology. Groups of galaxies in the large-scale structure of the Universe are observed in sky surveys, i.e. we have a distribution of particles in space although there are some differences with respect to the liquid. First of all, we know that these particles are not all identical; their mass and other characteristics that are not important dynamically are distinctive among themselves. In view of this, we could face problems when applying statistical mechanics on astronomical scales, but the issue of different masses is still an important difficulty to resolve first.

Our purpose is to obtain a force equation for a distribution of particles with different masses that will allow different magnitudes of interest to be related among themselves. Our result will apply directly to Newtonian gravity for a Universe in expansion.

The necessary condition for solving the problem is to derive information about: (i) the probabilities of different configurations and (ii) the Maxwell-Boltzmann equilibrium for classical particles, which is the case that I will solve. This approach was already considered by other authors for the largest scales in the Universe (e.g. Saslaw & Hamilton 1984; Saslaw 1985) and I also believe that this condition could be applicable in some gravitational systems and in some scales (not all though, see Betancort-Rijo 1988): for example, the distribution of galaxies may be a good candidate to consider relaxed in which we will develop an example of the expression derived here in Sect. VIII. Nevertheless, there might be other gravitational systems for which this approach may be valid.

Send offprint requests to: martinlc@iac.es

2. The probability of mass and position for a particle with a known mass function distribution and Boltzmann equilibrium.

HYPOTHESIS 1: *We have a grand canonical ensemble of N point-particles, interchangeable with an external reservoir, with positions $\boldsymbol{r}_1$,...,$\boldsymbol{r}_N$ momenta $\boldsymbol{p}_1$,..., $\boldsymbol{p}_N$ and masses m_1,...,m_N respectively. This means that the distribution of positions and momenta in the particles follows a Maxwell-Boltzmann distribution, i.e. they are relaxed.*

The probability of a configuration in positions and masses is $\mathcal{P}_N(\boldsymbol{r}_1,...,\boldsymbol{r}_N;m_1,...,m_N)$. First, we define a general distribution of masses, and then, once we have the mass of each particle established a priori, the probability of a particle occupying a position depends on its mass:

$$\mathcal{P}_N(\boldsymbol{r}_1,...,\boldsymbol{r}_N;m_1,...,m_N)$$

$$= e^{\beta\mu N}P_N(\boldsymbol{r}_1,...,\boldsymbol{r}_N/m_1,...m_N)\Phi_N(m_1,...,m_N), \tag{1}$$

where μ is the chemical potential, β is a constant, Φ_N is the mass distribution function, and P_N the position distribution function which also depends on the masses as parameters (the probability of a configuration of positions is conditioned by the distribution of masses; the slash stands for "conditioned"). We have decomposed the probability $\mathcal{P}_N$ into the product of three probabilities: first, the probability of having N particles in a grand canonical ensemble (see for example Saslaw 1985, ch. 34); secondly, the probability of a mass configuration, Φ_N; and, finally, once we know the mass of each particle, the probability of the positions configuration. In the following subsections I obtain P_N and Φ_N.

2.1. *Probability of positions*

We have assumed that the number of particles is discrete and that the probability of that they be configured in a certain way is given by a function $P_N(\boldsymbol{r}_1,...,\boldsymbol{r}_N/m_1,...,m_N)$.

To analyse the form of P_N an assumption is needed on how position space (or the phase space that includes it) is populated, and this can be carried out by assuming a state of thermodynamic equilibrium, which was our first hypothesis. The Boltzmann distribution is characterized by a configuration probability proportional to the negative exponential of the Hamiltonian of the system, and the Hamiltonian is the sum of two terms, one depending on the momenta of the particles and other on the positions of the particles.

The Boltzmann probability distribution function for a fixed time (see for example Stanley 1971) is

$$P_N(\boldsymbol{r}_1,...,\boldsymbol{r}_N;\boldsymbol{p}_1,...,\boldsymbol{p}_N/m_1,...,m_N)$$

$$\propto e^{-\beta\mathcal{H}(r_1,...,r_N;p_1,...,p_N/m_1,...,m_N)}$$

$$= e^{-\beta U(r_1,...,r_N/m_1,...,m_N)}e^{-\frac{1}{2}\beta\sum_{i=1}^N\frac{p_i^2}{m_i}}, \tag{2}$$

where $\mathcal{H}$ is the Hamiltonian of the gravitational system, U the potential energy and $\frac{1}{2}\sum_{i=1}^N\frac{p_i^2}{m_i}$ the kinetic energy. We integrate over the momentum space $\boldsymbol{p}_i$, thus obtaining the probability for the positions space configuration:

$$P_N(\boldsymbol{r}_1,...,\boldsymbol{r}_N/m_1,...,m_N) \propto e^{-\beta U(r_1,...,r_N;m_1,...,m_N)}$$

$$\times\int d\boldsymbol{p}_i...\int d\boldsymbol{p}_N e^{-\frac{1}{2}\beta\sum_{i=1}^N\frac{p_i^2}{m_i}}. \tag{3}$$

The masses m_i and β are constant so the integrals in (3) are constant and

$$P(\boldsymbol{r}_1,...,\boldsymbol{r}_N/m_1,...,m_N)_N \propto e^{-\beta U(r_1,...,r_N/m_1,...,m_N)}. \tag{4}$$

2.2. *Probability of masses*

Obviously, if we have to take into account the difference of masses between particles, we must first derive the mass distribution. The usual tool for doing this is the mass distribution function for a particle, $\phi(m)$, i.e. the probability of a particle with mass between m and $m+dm$ is $\phi(m)dm$. Actually, ϕ represents an average over space of positions of the mass distribution. We could consider the distribution of mass to vary at different positions, i.e. the centre of a galaxy cluster with respect to another position, an effect which should indeed be taken into account. But this is wrong. Remember that by means of (1), we first chose the masses of the particles according to their probabilities, and then we assigned a position to the particle conditional on having a previously established mass.

So the probability of N particles having masses m_1,...,m_N, respectively, is

$$\Phi_N(m_1,...,m_N) \propto \phi(m_1)...\phi(m_N). \tag{5}$$

2.3. *Normalized probability*

Expressions (1), (4) and (5), once normalized, lead to

$$\mathcal{P}_N(m_1,...,m_N;U(\boldsymbol{r}_1,...,\boldsymbol{r}_N/m_1,...,m_N))$$

$$= e^{\beta\mu N}\phi(m_1)...\phi(m_N)\frac{e^{-\beta U(r_1,...,r_N/m_1,...,m_N)}}{Z}, \tag{6}$$

and the partition function of this canonical mass-dependent distribution is given by

$$Z = \sum_{N=0}^{\infty} e^{\beta\mu N}\int dm_1...\int dm_N\int d\boldsymbol{r}_1...d\boldsymbol{r}_N$$

$$\times\phi(m_1)...\phi(m_N)e^{-\beta U(r_1,...,r_N/m_1,...,m_N)}. \tag{7}$$

3. Correlation functions

The mathematical tools that allow important filtered information to be extracted from the distribution of particles are the correlation functions. These will enable us to ascertain whether the distribution is Poissonian, or how large the difference is with respect to a Poissonian distribution.

Hereafter, I define certain functions that contain the symbol $<(\cdots)>$ as meaning an average of the quantity $(\cdots)$. There are two different ways of calculating these averages: (i) by volume, and (ii) statistically-mechanically. I now describe the two ways:

(i) In this case, the average is calculated according to the expression

$$<(\cdots)>=\frac{1}{V}\int_V d\boldsymbol{r}(\cdots), \tag{8}$$

where $(\cdots)$ represents the quantity whose average we wish to obtain. It is clear that we are calculating a volume average. This is the method that describes the distribution macroscopically. The observational correlations are extracted with this algorithm.

(ii) In this case, the average calculation is conducted by means of the expression

$$<(\cdots)>=\sum_{N=0}^{\infty}\int dm_1...dm_N\int d\boldsymbol{r}_1...d\boldsymbol{r}_N(\cdots)$$

$$\times\mathcal{P}_N(\boldsymbol{r}_1,...,\boldsymbol{r}_N;m_1,...,m_N), \tag{9}$$

where $(\cdots)$ is the same as in (i). This method appeals to the microscopic properties of the physical system.

The definitions that follow include an expansion with the second averaging method.

3.1. Particle-particle two-point correlation function

The density of the particles is $n(\boldsymbol{r})=\sum_{i=1}^N\delta(\boldsymbol{r}-\boldsymbol{r}_i)$.

The two-point correlation function for particles without autocorrelation ($i\neq j$ in the following expression) is, from (6) and (9),

$$<n(\boldsymbol{r})n(\boldsymbol{r'})>=\sum_{N=0}^{\infty}e^{\beta\mu N}\int dm_1...\int dm_N\int d\boldsymbol{r}_1...d\boldsymbol{r}_N$$

$$\times n(\boldsymbol{r})n(\boldsymbol{r'})\phi(m_1)...\phi(m_N)\frac{e^{-\beta U(r_1,...,r_N/m_1,...,m_N)}}{Z}$$

$$=\sum_{N=0}^{\infty}e^{\beta\mu N}\int dm_1...\int dm_N\int d\boldsymbol{r}_1...d\boldsymbol{r}_N$$

$$\times\left(\sum_{i,j=1;i\neq j}^{N}\delta(\boldsymbol{r}-\boldsymbol{r}_i)\delta(\boldsymbol{r'}-\boldsymbol{r}_j)\right)$$

$$\times\phi(m_1)...\phi(m_N)\frac{e^{-\beta U(r_1,...r_N/m_1,...m_N)}}{Z}. \tag{10}$$

In order to simplify the notation, the mass dependence of U will not be indicated in what follows. A straightforward calculation shows that

$$<n(\boldsymbol{r})n(\boldsymbol{r'})>=\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{N(N-1)}{Z}\int dm_1...\int dm_N$$

$$\times\int d\boldsymbol{r}_3...d\boldsymbol{r}_N\phi(m_1)...\phi(m_N)e^{-\beta U(r,r',r_3,...,r_N)}. \tag{11}$$

If we assume isotropy, the function only depends on $|\boldsymbol{r}-\boldsymbol{r'}|$; I denote this as $<nn>(|\boldsymbol{r}-\boldsymbol{r'}|)$.

3.2. Mass-mass two-point correlation function

The mass density is $\rho(\boldsymbol{r})=\sum_{i=1}^N m_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)$. The two-point correlation function for mass without autocorrelation is

$$<\rho(\boldsymbol{r})\rho(\boldsymbol{r'})>=\sum_{N=0}^{\infty}e^{\beta\mu N}\int dm_1...\int dm_N\int d\boldsymbol{r}_1...d\boldsymbol{r}_N$$

$$\times\rho(\boldsymbol{r})\rho(\boldsymbol{r'})\phi(m_1)...\phi(m_N)\frac{e^{-\beta U(r_1,...,r_N)}}{Z}$$

$$=\sum_{N=0}^{\infty}e^{\beta\mu N}\int dm_1...\int dm_N\int d\boldsymbol{r}_1...d\boldsymbol{r}_N$$

$$\times\left(\sum_{i,j=1;i\neq j}^{N}m_im_j\delta(\boldsymbol{r}-\boldsymbol{r}_i)\delta(\boldsymbol{r'}-\boldsymbol{r}_j)\right)\phi(m_1)...\phi(m_N)$$

$$\times\frac{e^{-\beta U(r_1,...,r_N)}}{Z}=\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{N(N-1)}{Z}\int dm_1...\int dm_N$$

$$\times\int d\boldsymbol{r}_3...d\boldsymbol{r}_N m_1m_2\phi(m_1)...\phi(m_N)e^{-\beta U(r,r',r_3,...,r_N)}. \tag{12}$$

With isotropy, the function depends on $|\boldsymbol{r}-\boldsymbol{r'}|$.

3.3. Mass-particle two-point correlation function

With the same philosophy in mind, we define other functions as the two-point correlation function mass-particle:

$$<\rho(\boldsymbol{r})n(\boldsymbol{r'})>=\sum_{N=0}^{\infty}e^{\beta\mu N}\int dm_1...\int dm_N\int d\boldsymbol{r}_1...d\boldsymbol{r}_N$$

$$\times\left(\sum_{i,j=1;i\neq j}^{N}m_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)\delta(\boldsymbol{r'}-\boldsymbol{r}_j)\right)$$

$$\times\phi(m_1)...\phi(m_N)\frac{e^{-\beta U(r_1,...,r_N)}}{Z}=\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{N(N-1)}{Z}$$

$$\times\int dm_1...\int dm_N\int d\boldsymbol{r}_3...d\boldsymbol{r}_N m_1\phi(m_1)...\phi(m_N)$$

$$\times e^{-\beta U(r,r',r_3,...,r_N)}. \quad (13)$$

With isotropy, the function depends on $|\boldsymbol{r}-\boldsymbol{r}'|$.

3.4. *Mass-mass-particle three-point correlation function*

This function is defined as follows:

$$<\rho(\boldsymbol{r})\rho(\boldsymbol{r}')n(\boldsymbol{r}'')>$$

$$=\sum_{N=0}^{\infty}e^{\beta\mu N}\int dm_1...\int dm_N\int d\boldsymbol{r}_1...d\boldsymbol{r}_N$$

$$\times\left(\sum_{\substack{i,j,k=1\\ i\neq j,j\neq k,k\neq i}}^{N}m_im_j\delta(\boldsymbol{r}-\boldsymbol{r}_i)\delta(\boldsymbol{r}'-\boldsymbol{r}_j)\delta(\boldsymbol{r}''-\boldsymbol{r}_j)\right)$$

$$\times\phi(m_1)...\phi(m_N)\frac{e^{-\beta U(r_1,...,r_N)}}{Z}$$

$$=\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{N(N-1)(N-2)}{Z}\int dm_1...\int dm_N\int d\boldsymbol{r}_4...$$

$$\times d\boldsymbol{r}_N m_1m_2\phi(m_1)...\phi(m_N)e^{-\beta U(r,r',r'',r_4,...,r_N)}. \quad (14)$$

With isotropy, this function depends on three variables: $|\boldsymbol{r}-\boldsymbol{r}'|$, $|\boldsymbol{r}-\boldsymbol{r}''|$ and $|\boldsymbol{r}'-\boldsymbol{r}''|$.

4. A force equation for homogeneous, isotropic distributions in Boltzmann equilibrium

In this section, I derive the force equation and then introduce a Newtonian form for the interaction.

The only trick is to apply the operator ∇_{r_1} over expression (11) and to develop the expression

$$\nabla_{r_1}<n(\boldsymbol{r}_1)n(\boldsymbol{r}_2)>=\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{N(N-1)}{Z}\int dm_1...$$

$$\times\int dm_N\int d\boldsymbol{r}_3...d\boldsymbol{r}_N\phi(m_1)...\phi(m_N)e^{-\beta U(r_1,...,r_N)}$$

$$\times(-\beta)\nabla_{r_1}U(\boldsymbol{r}_1,...,\boldsymbol{r}_N). \quad (15)$$

Hypothesis 2: *The interaction among particles is between pairs of particles, and is proportional to their masses, as a function of their distance (it is a central force).*

By this hypothesis, the total potential energy is the sum of the potential energies between pairs of particles ($U(\boldsymbol{r}_1,...,\boldsymbol{r}_N)=\sum_{i<j}V_{ij}(\boldsymbol{r}_i-\boldsymbol{r}_j)$). Hence,

$$\nabla_{r_1}<n(\boldsymbol{r}_1)n(\boldsymbol{r}_2)>=\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{-\beta N(N-1)}{Z}\int dm_1...$$

$$\times\int dm_N\int d\boldsymbol{r}_3...d\boldsymbol{r}_N\phi(m_1)...\phi(m_N)e^{-\beta U(r_1,...,r_N)}$$

$$\times\left[\nabla_{r_1}V_{12}(\boldsymbol{r}_1-\boldsymbol{r}_2)+\sum_{i=3}^{N}\nabla_{r_1}V_{1i}(\boldsymbol{r}_1-\boldsymbol{r}_i)\right]. \quad (16)$$

This becomes

$$\nabla_{r_1}<n(\boldsymbol{r}_1)n(\boldsymbol{r}_2)>=\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{-\beta N(N-1)}{Z}\int dm_1...$$

$$\times\int dm_N\nabla_{r_1}V_{12}(\boldsymbol{r}_1-\boldsymbol{r}_2)\int d\boldsymbol{r}_3...d\boldsymbol{r}_N\phi(m_1)...\phi(m_N)$$

$$\times e^{-\beta U(r_1,...,r_N)}+\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{-\beta N(N-1)(N-2)}{Z}$$

$$\times\int dm_1...\int dm_N\int d\boldsymbol{r}_3\nabla_{r_1}V_{13}(\boldsymbol{r}_1-\boldsymbol{r}_3)$$

$$\times\int d\boldsymbol{r}_4...d\boldsymbol{r}_N\phi(m_1)...\phi(m_N)e^{-\beta U(r_1,...,r_N)}. \quad (17)$$

Hypothesis 3: *We assume homogeneity and isotropy.*

This means that $<n(\boldsymbol{r}_i)n(\boldsymbol{r}_j)>$ only depends on distance $r_{ij}\equiv|\boldsymbol{r}_i-\boldsymbol{r}_j|$ and $V(\boldsymbol{r})=V(r)$. The operator ∇_{r_1} can be expanded and thus (I note $<n_1n_2>$ instead of $<nn>$ to avoid confusion; the same with $<\rho\rho>$ and $<\rho\rho n>$ in the following equations)

$$r_{12}\frac{\partial<n_1n_2>(r_{12})}{\partial r_{12}}=\sum_{N=0}^{\infty}e^{\beta\mu N}\frac{-\beta N(N-1)}{Z}$$

$$\times\int dm_1...\int dm_N[r_{12}\frac{\partial V_{12}(r_{12})}{\partial r_{12}}]\int d\boldsymbol{r}_3...d\boldsymbol{r}_N$$

$$\times\phi(m_1)...\phi(m_N)e^{-\beta U(r_1,...,r_N)}+\sum_{N=0}^{\infty}e^{\beta\mu N}$$

$$\times\frac{-\beta N(N-1)(N-2)}{Z}\int d\boldsymbol{r}_3\int dm_1...\int dm_N$$

$$\times\left[r_{12}\cos(\boldsymbol{r}_{12},\boldsymbol{r}_{13})\frac{\partial V_{13}(r_{13})}{\partial r_{13}}\right]$$

$$\times\int d\boldsymbol{r}_4...d\boldsymbol{r}_N\phi(m_1)...\phi(m_N)e^{-\beta U(r_1,...,r_N)}. \quad (18)$$

I rename $r = r_{12}$, $s = r_{13}$, $\theta \equiv$ angle between $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_{13}$ and assume that $V_{ij}(r) = m_i m_j v(r)$ (valid in the light of Hypothesis 2) and substitute some expressions using (12) and (14), taking into account the isotropy assumed in Hypothesis 3:

$$\frac{-1}{\beta}\frac{\partial < n_1 n_2 > (r)}{\partial r} = \frac{\partial v(r)}{\partial r} < \rho\rho > (r)$$

$$+2\pi \int_0^\pi d\theta \sin\theta\cos\theta \int_0^\infty ds\ s^2$$

$$\times \frac{\partial v(s)}{\partial s} < \rho_1\rho_2 n_3 > (r, s, \sqrt{r^2+s^2-2rs\cos\theta}). \qquad (19)$$

This is the force equation. It gives us a relationship between different correlation functions in a distribution that follows our hypothesis and the force that is represented by means of v.

5. Newtonian interaction and expansion

5.1. Newtonian interaction

Before proceeding, I will comment on some aspects regarding the Newtonian case in particular. Certain problems are associated with the application of principles such as thermodynamics and statistical mechanics in groups of particles with r^{-2}-type force (Taff 1985): some divergences are found, and there is non-saturation of gravitational forces (Levy-Leblond 1969). Some authors take the view that there can be no rigorous basis for applying statistical mechanics in such a system (Fisher & Ruelle 1966). This result is not a consequence of the r^{-2}-type force but rather of its unshielded character (see Dyson & Lenard 1967 for a discussion of the electrostatic case).

In any event, one avoids the non-self-consistency of the problem by truncating the integral limits at a finite radius, or by cancelling the correlation functions that fall under a given lower unit, assuming points with negligible volume as is the case in real physical problems. In my opinion, N-body systems exist in nature under Newtonian gravitational forces, and to invoke a distribution of particles in such systems is not necessarily inconsistent. The possible infinities that appear in some expressions are only mathematical problems which are not present in nature and can be solved once we give our data the conditions that will accommodate our physical reality to a mathematical model (I mean to avoid the infinite proximity of particles by means of a cut-off, etc.). We can look at it from another point of view: a statistical thermodynamical system in equilibrium cannot achieve singular states with infinities, except as a set of zero measurements, because the probability of getting a singular state, with a very small distance between some particles, would take an extremely long time, as if approaching a singularity.

Expression (19) has been obtained regardless of the force type, so it possesses a general validity. Now, when we introduce a Newtonian gravitational force the expression continues to be valid. If an infinity appears in the next expression it is only a question of truncating the integrals or selecting the best correlation function that does not produce divergences. When we take the cut-off, we neglect the probabilities near the singularity, a set with dimensions greater than zero but small enough. The introduction of cut-offs will make the results dependent upon the details of the regularization, so the selection of the cut-offs must have a physical basis.

Hypothesis 4: *The interaction between pairs of particles is the Newtonian gravity force.*

This hypothesis obviously includes Hypothesis 2. With a Newtonian potential: $V_{ij}(\boldsymbol{r}_k - \boldsymbol{r}_l) = -\frac{G m_i m_j}{r_{kl}}$. Then,

$$\frac{\partial v(r)}{\partial r} = \frac{G}{r^2}.$$

We set $C \equiv (\beta G)^{-1}$. This leads to

$$-C\frac{\partial < n_1 n_2 > (r)}{\partial r} = \frac{1}{r^2} < \rho_1\rho_2 > (r)$$

$$+2\pi \int_0^\pi d\theta \sin\theta\cos\theta \int_0^\infty ds$$

$$\times < \rho_1\rho_2 n_3 > (r, s, \sqrt{r^2+s^2-2rs\cos\theta}), \qquad (20)$$

which gives us a relationship among certain correlations of the distribution and C.

The proof that the system is valid for achieving thermodynamic equilibrium can be found in Lieb & Lebowitz (1973), where a general Coulombian system is considered. Nevertheless, this does not imply that all Newtonian gravitational systems are in equilibrium.

5.2. Effects of the expansion of the Universe

When we consider the galaxy distribution in the large-scale structure of the Universe, we must bear in mind the expansion, so:

Hypothesis 5: *The system of particles is distributed over a space in expansion.*

We must use comoving coordinates in order to maintain the zero peculiar velocity as the most probable one (because the above formulation gives the probability as proportional to e^{-p}, where p is the momentum). We consider the proper motions, not the background: the kinetic energy derived from peculiar velocities and the comoving potential energy.

It is derived in Saslaw & Fang (1996) that consideration of the expansion is equivalent to taking into account the gravitational effects of the local fluctuating part of the

density field when we use comoving coordinates, i.e. we should substract the mean density from the density field. Since the mean density does not depend on r or θ and the integral $\int_0^\pi d\theta \sin\theta\cos\theta = 0$, equation (20) is modified by simply replacing $<\rho_1\rho_2>(r)$ by $<\rho_1\rho_2>(r)-<\rho>^2$:

$$-C\frac{\partial <n_1 n_2>(r)}{\partial r} = \frac{1}{r^2}(<\rho_1\rho_2>(r)-<\rho>^2)$$

$$+2\pi\int_0^\pi d\theta \sin\theta\cos\theta\int_0^\infty ds$$

$$\times <\rho_1\rho_2 n_3>(r,s,\sqrt{r^2+s^2-2rs\cos\theta}). \quad (21)$$

6. The meaning of β

We know the meaning of β in statistical physics; it is equal to $\frac{1}{K_{\rm B}T}$, where $K_{\rm B}$ is Boltzmann's constant, and T is the absolute temperature. In the large-scale (or any other) structure we can establish a similar meaning by developing a kinetic theory, in which the particles are the galaxies themselves (or other particles).

If we integrate expression (2) over position space, we get

$$\mathcal{P}(\boldsymbol{p}_1,...,\boldsymbol{p}_N;m_1,...m_N)$$

$$\propto \phi(m_1)...\phi(m_N)e^{-\frac{1}{2}\beta\sum_{i=1}^N\frac{p_i^2}{m_i}}. \quad (22)$$

The mean value of the peculiar velocities for all clustering galaxies is

$$\overline{v} = \frac{\int dm\phi(m)\int d\boldsymbol{p}\frac{p}{m}e^{-\frac{1}{2}\beta\frac{p^2}{m}}}{\int dm\phi(m)\int d\boldsymbol{p}e^{-\frac{1}{2}\beta\frac{p^2}{m}}}$$

$$= \frac{\int dm\frac{\phi(m)}{m}\frac{8\pi m^2}{\beta^2}}{\int dm\phi(m)\left(\frac{2\pi m}{\beta}\right)^{3/2}} = \left(\frac{8}{\pi\beta}\right)^{1/2}\frac{\int dm\phi(m)m}{\int dm\phi(m)m^{3/2}}. \quad (23)$$

We can also express this as

$$\overline{v} = \left(\frac{8GC}{\pi\mu}\right)^{1/2}, \quad (24)$$

where $\mu = \left(\frac{\int dm\phi(m)m}{\int dm\phi(m)m^{3/2}}\right)^{-2}$ and C represent the same constant as in the force equation for the Newtonian case. Hence, note that the parameter C, which we could obtain in the equality (21), will give information about the velocity field of the particles, or, vice-versa, we could obtain C for (21) from the mean velocity.

The parameter β is even more directly related to the kinetic energy. An analogous derivation leads to

$$E_{\rm kin} = \frac{3N}{2\beta} = \frac{3}{2}N\ G\ C, \quad (25)$$

a fundamental result of statistical mechanics.

7. How to obtain the correlations from the distribution

To this end, we have to use eq. (8) as defined previously. However, when we have a discrete number of points instead of a continuum distribution, perhaps it might be better to use other equivalent expressions.

When homogeneity is given, one method discussed by Rivolo (1986) is to use the estimator

$$<n_1 n_2>(r) = \frac{<n>}{N}\sum_{i=1}^N\frac{N_i(r)}{V_i(r)}, \quad (26)$$

where $N_i(r)$ is the number of particles lying in a shell of thickness δr from the ith particle, $V_i(r)$ is the volume of the shell lying within the sample volume and $<n>$ is the average density in a macroscopically homogeneous system ($<n>$ is independent of position).

The evaluation of $<\rho_1\rho_2>$ and $<\rho n>$ would be:

$$<\rho_1\rho_2>(r) = \frac{<\rho>}{M}\sum_{i=1}^N\frac{m_i M_i(r)}{V_i(r)} \quad (27)$$

and

$$<\rho_1 n_2>(r) = \frac{<n>}{M}\sum_{i=1}^N\frac{m_i N_i(r)}{V_i(r)}, \quad (28)$$

where $M = \sum_{i=1}^N m_i$ and $M_i(r)$ is the mass lying in a shell of thickness δr from the ith particle. Knowledge of m_i is sometimes problematic and its solution differs in each case. In the case of galaxies or stars information is required about the mass-luminosity relationship as well as data on magnitudes and distances. This could also be provided with a knowledge of the mass distribution function according to different zones, ($\phi(m)$ would be the average of the different mass distribution functions in the entire space) and by randomly assigning a mass m_i to each particle following these distribution functions that would give us not the real mass distribution but an equivalent one. In any case, the problem of assigning mass is different in each case and generally requires tailor-made solutions for each one.

The three-point correlation function is also obtainable by counting groups of three particles with different distances between them. Usually, in the isotropic case, this is approximated as a function of different two-point correlation functions. For example, in liquid theory the so-called superposition approximation is commonplace (see for example March & Tosi 1976 and first formulation of it in Kirkwood 1935), and applied to the mass-mass-particle case would take the form

$$<\rho_1\rho_2 n_3>(r,s,t)$$

$$= \frac{<\rho_1\rho_2>(r)<\rho_1 n_2>(s)<\rho_2 n_3>(t)}{<\rho>^2<n>}. \quad (29)$$

From the large-scale distribution of galaxies in the Universe, we can extract statistical information (as in Saunders et al. 1991). When we observe the projected distribution onto a 2-dimensional surface, i.e. we do not know the distance of the objects, we can obtain the correlation on the 2-dimensional surface (angular correlation) and relate it to the 3-dimensional distribution correlations by means of Limber's equation (Peebles 1980). Maddox et al. (1990) obtained the two-point angular correlation function on large scales, so we can derive $< n_1 n_2 >$. To assign the masses in order to achieve $< \rho_1 \rho_2 >$ and $< \rho_1 n_2 >$ is again troublesome and depends on the data available on the masses of the galaxies.

For the mass of galaxies in the large-scale structure, future data will become available with the help of the DENIS project (Mamon 1995), a near-infrared sky survey. The near-infrared is thought to be a better tracer of the stellar mass in the galaxies, and if the stellar mass content follows the total mass content (including dark matter), near-infrared surveys should be the best way of obtaining the distribution of matter in the Universe.

Also, on other astronomical scales we find point-particle distributions and we may obtain their correlations. Borgani et al. (1991) obtained the correlation functions for scales between 3 and 350 kpc, and this could be done for smaller scales as well. Another problem is posed by considering our hypotheses as valid, especially Hypothesis 1.

An application to any other distribution is also possible.

8. Application on the Large-scale distribution of galaxies

The actual application of these tools is quite difficult since real data are not easily available. To illustrate the way to proceed with this method, we apply the above theoretical results to a practical case with observational data, in the N-body system of the large scale distribution of galaxies in the Universe (see for example Peebles 1980; Borgani 1995) which is a homogeneous and isotropic distribution. A recent model, using equilibrium statistical mechanics as well as other considerations, was also developed in Pérez-Mercader et al. (1996).

We will assume the validity of the assumptions made in this paper and adopt an additional assumption here for the particular case of the use of (29), the superposition approximation, and biasing of the ρ fluctuation as proportional to the n fluctuation where the constant of proportionality depends on the scale:

$$\frac{\delta n}{n} = b(r)\frac{\delta\rho}{\rho}, \tag{30}$$

where b is the biasing parameter, dependent a priori on r, the distance between both particles. Astronomers usually make a stronger assumption by taking b to be constant, but I am not going to be so restrictive. I adopt a unit system where $< \rho > = < n >$ and set $< n_1 n_2 > (r) \equiv < n >^2 (1 + \xi(r))$. As a consequence,

$$< \rho_1 n_2 > = < n >^2 (1 + b^{-1}\xi(r)) \tag{31}$$

and

$$< \rho_1 \rho_2 > = < n >^2 (1 + b^{-2}\xi(r)). \tag{32}$$

Thus, (21) leads to:

$$-C\xi'(r) = \frac{b^{-2}(r)\xi(r)}{r^2}$$

$$+2\pi < n > \int_0^\infty ds[1 + b^{-1}(s)\xi(s) + b^{-2}(r)\xi(r)$$

$$+b^{-2}(r)b^{-1}(s)\xi(r)\xi(s)] \int_0^\pi d\theta \sin\theta\cos\theta$$

$$\times b^{-1}(\sqrt{r^2+s^2-2rs\cos\theta})\xi(\sqrt{r^2+s^2-2rs\cos\theta}). \tag{33}$$

This expression resembles the one given by Peebles (1976) for $b = 1$, the cosmic virial theorem, but with further generality because expression (33) takes into account that $< n_1 n_2 > (r) \neq < \rho_1 \rho_2 > (r)$.

An observational ξ was achieved by Groth & Peebles (1977):

$$\xi(r) = \left(\frac{r}{r_0}\right)^{-\gamma} \quad \text{for } 0.3h^{-1}\ \text{Mpc} < r < 10h^{-1}\ \text{Mpc} \tag{34}$$

where $r_0 = 4.1h^{-1}$ Mpc and $\gamma = 1.77$. The parameter ξ is negligible for $r > 10h^{-1}$ Mpc, so it is taken as zero. Also a cut-off is taken for $r < 0.3h^{-1}$ Mpc because separated galaxies cannot be at distances less than this cut-off (if two galaxies have a distance between themselves less than this then they are considered as only one galaxy), and this ensures convergence of the integral in (33).

In order to estimate the behaviour of $b(r)$, we introduce it as a power-law dependence, such that

$$b(r) = \left(\frac{r}{r_*}\right)^{\kappa} \tag{35}$$

and fit the best values of κ and r_* to solve (33). The constants are derived from observational data: $< n > = 0.02$ Mpc^{-3} (Allen 1973), $h = 0.60$ (the actual value of the Hubble constant in units of 100 km s^{-1} Mpc^{-1}) and $\overline{v} \sim 100$ km s^{-1} (Allen 1973).

If we now wish to obtain C, according to (24), we must convert G from our units (time unit: second; length unit: 1 h^{-1} Mpc; mass unit: the average mass of a galaxy, $\overline{m}_{\rm gal}$ in K‘kg) to the MKS system, multiplying by a factor of $(3.08 \times 10^{22})^{-3}\overline{m}_{\rm gal}$. Thus, denoting by $G_{\rm MKS}$ the value of G in the MKS system, we have

$$\overline{v} = \left(\frac{8G_{\rm MKS}(3.08\times10^{22})^{-3}\overline{m}_{\rm gal}C}{\pi}\right)^{1/2} h^{-1}\ \text{Mpc s}^{-1}, \tag{36}$$

so with $G_{\rm MKS} = 6.672 \times 10^{11}$, $M_\odot = 1.99 \times 10^{30}$ kg and 1 Mpc $= 3.08 \times 10^{19}$ km

$$C \approx \frac{3.45 \times 10^{13}\ M_\odot}{\overline{m}}. \tag{37}$$

This, together with an estimate of $\overline{m} \approx 8 \times 10^{10}\ M_\odot$ (Allen 1973), gives us

$$C \approx 4.3 \times 10^2. \tag{38}$$

The introduction of all these data into (33) and some calculations lead to the following expression for the allowed r values:

$$9.25 \times 10^3 r^{-2.77} - 12.1 r_*^{2\kappa} r^{-3.77-2\kappa} = \frac{0.291 r_*^\kappa}{r}$$

$$\times \int_0^{r+10} \frac{ds}{s}[1 + 12.1 r_*^\kappa \left\{ \begin{array}{ll} -1 & s<0.3 \\ s^{-1.77-\kappa} & 0.3<s<10 \\ 0 & s>10 \end{array} \right\}$$

$$+12.1 r_*^{2\kappa} r^{-1.77-2\kappa} + 83.1 r_*^{3\kappa} \left\{ \begin{array}{ll} -1 & s<0.3 \\ s^{-1.77-\kappa} & 0.3<s<10 \\ 0 & s>10 \end{array} \right\}]$$

$$\times [\frac{1}{2.23-\kappa} \left(\left\{ \begin{array}{ll} 0.3^{2.23-\kappa} & |r-s|<0.3 \\ |r-s|^{2.23-\kappa} & |r-s|>0.3 \end{array} \right\} - (r+s)^{2.23-\gamma} \right)$$

$$-\frac{r^2+s^2}{0.23-\kappa} \left(\left\{ \begin{array}{ll} 0.3^{0.23-\kappa} & |r-s|<0.3 \\ |r-s|^{0.23-\kappa} & |r-s|>0.3 \end{array} \right\} - (r+s)^{0.23-\gamma} \right)]. \tag{39}$$

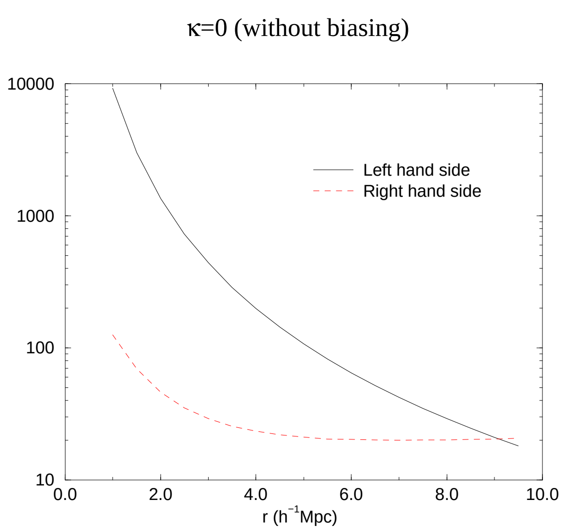

Fig. 1. Comparison between the left- and right-hand side of expression (39) for the non-biasing case.

If non-biasing, i.e. $b = 1$, were assumed for all scales, we would obtain the results plotted in Fig. 1, so biasing is neccesary.

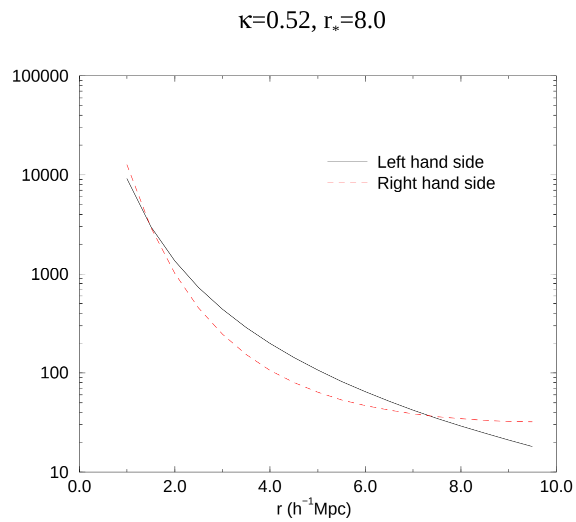

Fig. 2. Comparison between the left- and right-hand side of expression (39) for best fit of κ and r_*.

The best fit, calculated numerically, is shown in Fig. 2, where left- and right-hand sides of expression (39) for parameters $\kappa = 0.52$ and $r_* = 8.0\ h^{-1}$ Mpc are plotted. With these parameters we get

$$b(r) = \left(\frac{r}{8.0\ h^{-1}\ \mathrm{Mpc}} \right)^{0.52}, \tag{40}$$

or, using a different expression

$$\xi_{\rho\rho} = b^{-2}(r)\xi(r) = \left(\frac{r}{5.3 h^{-1}\ \mathrm{Mpc}} \right)^{-2.81} \tag{41}$$

in the range above defined for ξ, where $<\rho\rho>(r) = <n>^2 (1 + \xi_{\rho\rho}(r))$. The obtained power law for mass correlation, "the -2.8 power law" is different from that of objects' correlation, "the -1.8 power law". Of course, the left and hand sides of (39) were not expected to agree perfectly and in fact they do not (Fig. 2) because we assumed a b dependence, a power law, which might not be very realistic. A worthwhile result is that a deviation from $b = 1$ is necessary to get an agreement between the left- and right-hand sides of (39), and the solution must be close to $b(r) = 0.34 r^{0.52}$. This result increases the information about the mass-mass correlation function (see Peebles 1980), which was unknown till now. If this result is true and is verified by alternative methods, it would give us important information about the "dynamics" in the large scale distribution of matter in the Universe.

The dependence of the outcome on the cut-off value is not negligible, but neither is it very pronounced. Some numerical results were obtained with other values of the cut-off, and the qualitative result does not differ too much: a cut-off at $r = 0.1\ h^{-1}$ Mpc instead of $r = 0.3\ h^{-1}$ Mpc,

gives us $\kappa = 0.47$ instead of 0.52, and $r_* = 8.1\ h^{-1}$ Mpc instead of $r_* = 8.0\ h^{-1}$ Mpc.

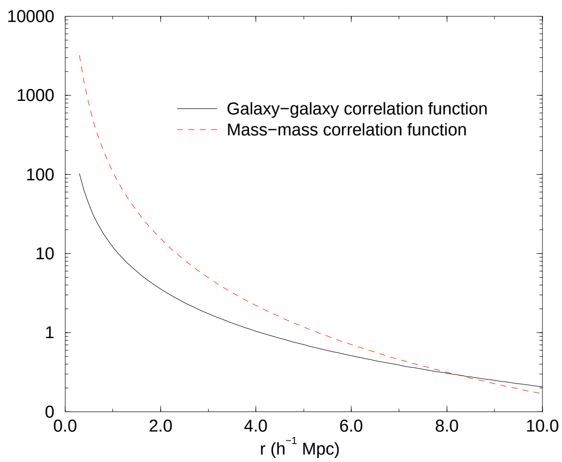

Fig. 3. The galaxy-galaxy correlation function, ξ, derived from observational data and $\xi_{\rho\rho}$, mass-mass correlation function, derived from ξ through the force equation explained in this paper.

The direct consequence of this is that mass is more correlated than the objects (Fig. 3). This means that dark matter must exist near galaxies and clusters of galaxies to increase the density contrast for short distances from an object (dark matter in form of discrete unseen galaxies, for instance dwarfs galaxies), unless the most massive galaxies are clumped together or there is some other solution, but something must explain the difference in both correlation functions for galaxies and mass. Deviations from equilibrium might also be responsible for part of the deviation attributed to biasing, as far as it is only an approximation. I am unware of how this may affect my results, essentially because the extent of this deviation is unknown, though I expect it to be not very large. A large departure from equilibrium would produce a fast evolution of the distribution and the distributions of matter far from us would have different distributions. This latter is not observed for long-range correlations [1], so we must infer that the departure from equilibrium conditions cannot be too high unless perhaps for very small ranges. For typical scales in galaxy clusters, we also find strong evidence of equilibrium (Carlberg et al. 1997). This discussion is beyond the scope of this paper. In Saslaw & Hamilton (1984, their Sect. 6), we find further arguments in favour of this: "Gravitational N-body experiments give good agreement with the theory. This shows that even though an equilibrium theory may not explain the ultimate fate of galaxy clustering, it does provide a good description over extremely long time scales when the correlations are 'frozen out' by the expansion of the Universe...[T]he equilibrium theory seems to explain most of the results."

The main question is how equilibrium could be reached in a short lifetime of the Universe. Violent relaxation which enormously decreases the relaxation time is a possible solution. Indeed, Saslaw (1985) points out that this must be the mechanism that governs the system due to large-scale collective modes (see chapter 38 of Saslaw 1985) and Henriksen & Widrow (1997) makes numerical simulations achieving this. The consideration of a steady state in the large-scale structure or a relaxation included in the initial conditions of the large-scale structure dynamics, before the formation of the galaxies, are other possible explanations.

The deviation from equilibrium and the corrections to make to our equations to take these effects into account are topics for future papers. Further research is necessary in this area to render these results more accurate. With this example, we wanted to show the way of working with the expressions described in this paper.

Readers might ask why I have not used the force equation to derive the two point-correlation function. To do this, I would need a knowledge of the biasing first, I cannot derive both things at the same time. Since the two point correlation function is better known that the bias, I decided to apply the method as described above. I had previously done some calculations to derive the two-point correlation function assuming non-biasing and the result was not compatible with observations, so I rejected this hypothesis thinking that biasing is necessary as I obtained here.

9. Results, other applications and further commentaries

We have a numerical relationship between the distribution of galaxies in space, which is represented by the correlation functions, and the two-galaxy interaction, which is represented by the potential energy.

We have obtained the equality (21) that must be followed by the distributions under Hypotheses 1 to 5. Also, we have equality (19) for the correlation functions and any interaction force under Hypotheses 1, 2 and 3 (we could also obtain an expresion like this including the expansion of space by means of the method explained in the subsection dedicated to the expansion).

Equation (21) relates the distribution correlation with the mean velocity of the galaxies by means of C with (24), and I believe it will be useful for obtaining a parameter from others that are already known in the distribution: the mean velocity from a complete knowledge of the correla-

[1] Broadhurst et al. (1990) deduces from observational data a regular distribution of galaxies similar to the nearby structure up to distances of $\sim 1000\ h^{-1}$ Mpc (a back in time further away from $\sim 4\times10^9$ yr)

tion functions, or an unknown parameter in the correlation function from the rest of the data. We could even obtain more than one parameter: two or three (in my opinion, more than three are too many) that follow the equality between the right- and left-hand sides of equation (21).

When equality between the two-sides of equation (21) is unattainable, this will indicate that our hypotheses are unsuitable. Probably, the most doubtful hypothesis is the first, i.e. that of Boltzmann equilibrium, and it is possible to verify the relaxation using this equation. [2]

In a sufficiently evolved system, Boltzmann equilibrium is achieved because the particle-points are classical particles and the probability of a state in such a case is proportional to the number of different states for each particle that preserves the number of particles and the total energy (the reader is referred to any book dealing with the foundations of statistical mechanics, e.g. Tolman 1938). It is also true that after a long time, the systems become virialized, and many systems are known to be in these conditions, although not all of them.

Otherwise, equation (19), *the force equation*, is a tool for looking for the kind of interaction in a system following Hypotheses 1, 2 and 3 (as purposed in Goldman et al. 1992). Once we have derived all the parameters of the distribution we can fit a shape for V, a two-body interaction potential that fits the equality, with or without expansion. We could even obtain other unknown parameters (C for example).

In order to demonstrate what might be the caveats in the implementation of the force equation, I developed a real example in the previous section where it was used to infer information about the mass-mass correlation function from the galaxy-galaxy correlation function, and the average density and peculiar velocity in the large-scale distribution of galaxies in the Universe. Further improvements are necessary, both in the observations and the theoretical assumptions, to obtain an accurate result, but the method is at least capable of telling us that the mass is more correlated than the galaxies at short distances (Fig. 3) when we assume relaxation on scales greater than 1 h^{-1} Mpc.

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[2] The verification should also be carried out with other methods because there exists some possibility of a verification of the equality due to other causes than relaxation, but the non-verification of the equality implies directly the non-equilibrium when the other hypotheses are right.

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